



ISSN: 1859-3100

TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH  
**TẠP CHÍ KHOA HỌC**

KHOA HỌC GIÁO DỤC  
Tập 15, Số 5b (2018): 139-152

HO CHI MINH CITY UNIVERSITY OF EDUCATION  
**JOURNAL OF SCIENCE**

EDUCATION SCIENCE  
Vol. 15, No. 5b (2018): 139-152

Email: tapchikhoahoc@hcmue.edu.vn; Website: http://tckh.hcmue.edu.vn

## AN EPISTEMOLOGICAL ANALYSIS OF THE CONCEPT OF QUOTIENT GROUP

*Nguyen Ai Quoc*\*

Saigon University

Received: 20/3/2018; Revised: 06/4/2018; Accepted: 21/5/2018

### ABSTRACT

*In the history of group theory, the quotient group is considered to be fundamental to the study of groups. The concept's development of quotient group is closely linked with the abstraction of group theory. This paper presents an epistemological analysis of the history of the development and formation of the "standard definition" of quotient group, which helps determine the epistemological characteristics and the obstacles for students in learning the quotient group.*

**Keywords:** obstacle, quotient group, epistemological analysis.

### TÓM TẮT

#### ***Một phân tích tri thức luận về khái niệm nhóm thương***

*Trong lịch sử của lý thuyết nhóm, nhóm thương được xem là nền tảng cho nghiên cứu các nhóm. Sự phát triển khái niệm nhóm thương có liên quan mật thiết đến sự trừu tượng hóa của lý thuyết nhóm. Bài báo này, trình bày một phân tích tri thức luận lịch sử về sự phát triển và hình thành "định nghĩa chuẩn" của nhóm thương, từ đó xác định các đặc trưng tri thức luận của nhóm thương và một số chương ngại đối với sinh viên khi nghiên cứu về tri thức này.*

**Từ khóa:** chương ngại, nhóm thương, phân tích tri thức luận.

### 1. Problematization

#### ***1.1. The necessity of quotient group study***

Quotient group is a rather complicated concept, for it is the combination of group concept and concept of set of cosets with an equivalent relationship. In the history of group theory, concept of quotient group is considered the foundation of study on groups; this concept, however, was unknown to group theorists due to the close relation between its development and the abstraction of group theory. The concept of quotient group emerged relatively late in the history of Mathematics (the end of the 19th century), and it was not until 1889 was the standard definition by Holder recognized by the Mathematic community and it has been used ever since. Hence, epistemological study is essential, through which we could define the epistemological characteristics and certain obstacles related to quotient group.

\* Email: nguyenaq2014@gmail.com

### 1.2. *The difficulties of students in approach to the concept of quotient group*

In July, 2016, a direct interview survey on concept of quotient group was conducted among 8 students majored in Pedagogy of Mathematics in Saigon University and Dong Nai University. These students had finished the course in Abstract Algebra (60 periods). The aim of this survey is to find out the students' difficulties in learning the nature of the elements and operations of quotient group.

The question raised was, “ $G$  is given as a group, and  $N$  a normal subgroup of  $G$ . Please mention the relationship between:

- 1/ quotient group  $G/N$  and group  $G$
- 2/ the elements of  $G/N$  and those of  $G$
- 3/ operations in  $G/N$  and those in  $G$ .” (Nguyen A. Q., 2017).

The survey results have shown the three main difficulties during the interviews:

- a) the distinction between the elements of the quotient group and those of the original group
- b) the comprehension of nature of the elements and that of the operations of quotient group
- c) the realization of fundamental factors in the building of a quotient group.

Five out of 8 students cannot realize the nature of the elements of quotient group  $G/N$  as well as its operations. Two of the five regard quotient group as a subgroup of  $G$ . One student simply explains that quotient group  $G/N$  is also a group and its elements come from the original group. The other student takes quotient group  $G/N$  as “a set of right cosets”, which is  $GN$  as product of elements of  $G$  with elements of  $N$ , thus being the elements of  $G$ . In terms of operations, the second student reckons two operations in  $G/N$  and  $G$  are identical.

Another student considers quotient group  $G/N$  a subgroup or a subset of  $G$ , and  $G/N$  multiplied by  $N$  will be  $G$ .

The other two students associate the concept of quotient group with the concept of arithmetic quotient. The first of these two regards the quotient group  $G/N$  as “the division of group  $G$  by subgroup  $N$ ” and gives an example  $Z/3Z$  as “the group divided by 3”. The other student considers the quotient group a set of quotients from an element of the original group divided by an element of the subgroup.

### 1.3. *The necessity of epistemological analysis*

The identification of types of errors made by students in learning Mathematics and their causes is always the first task of Mathematics Didacticians before they could suggest solutions for the students to avoid those errors. According to Brousseau (1983, p. 171),

“Errors are not only the consequence of unknown, uncertainty, spontaneity, as what empiricists and behaviorists think, but also the consequence of the previous knowledge, which is somehow useful for the former learning process but incorrect or simply inappropriate for the acquisition of new knowledge. Errors of this type are not irregular or unexpected. They become obstacles. In the teacher’s or students’ activities, errors usually help to build the significance of knowledge gained by such subjects.”

An epistemological analysis of the history of the concept of quotient group in an attempt to define the characteristics and the epistemological obstacles has brought about satisfactory explanation for Mathematics students’s mentioned errors in the light of Mathematics Didactic. This is also the objective of the research delivered in this article.

An epistemological analysis of the history of a concept is a study of the past to discover the formation process of a knowledge, the problems involved, the obstacles, the leaps in conception facilitating the advent of knowledge.

An epistemological analysis of a knowledge can clarify:

- The conditions, the obstacles of the advent of science knowledge and the “progress” of knowledge. Then, the epistemological obstacles can be defined. They are obstacles well associated with an epistemological analysis of the developmental history of knowledge, the overcoming of which plays a decisive role in the process of a subject’s construction of knowledge.
- The meaning of knowledge, as well as the problems that the knowledge can help to solve.
- Certain conceptions may be associated with knowledge.

## **2. The concept of quotient group**

### **2.1. Subgroup**

The concept of subgroup is introduced by Hoang Xuan Sinh, Tran Phuong Dung (2003, p. 22) in the teaching material after some consideration of group structure and the stable part. The concept of subgroup is defined as follows:

“A stable part  $A$  of a group  $X$  is called subgroup of  $X$  if  $A$  with the restriction of the group operation makes a group.”

### **2.2. Normal groups**

To bring about the concept of normal subgroups, Hoang Xuan Sinh, Tran Phuong Dung (2003, p. 31) addresses some of the constraints related to normalized subgroups such as:

- Equivalence relation on a subset;
- Concepts of left cosets and of right coset.

The concept of a normal subgroup is defined as follows:

“A subgroup  $A$  of a group  $X$  is called normal subgroup if and only if  $x^{-1}ax \in A$  for every  $a \in A$  and  $x \in X$ .”

### 2.3. Quotient group

After introducing the concept of normal subgroups, Hoang Xuan Sinh, Tran Phuong Dung (2003, p. 32) defines quotient groups by the theorem follows:

“If  $A$  is a normal subgroup of a group  $X$ , then:

i/ The rule for correspondence pair  $(xA, yA)$ , the left class  $xyA$  is a mapping from  $X/A \times X/A$  to  $X/A$ ;

ii/  $X/A$  with binary operations  $(xA, yA) \rightarrow xyA$  is a group, called the quotient group of  $X$  on  $A$ .”

### 2.4. Standard definition of quotient group

The definition of the quotient group which is given in Rose's textbooks (1978, pp. 42–43), Herstein (1975, pp. 51–52) and Macdonald (1968, pp. 56–57), called the "standard" modern definition as follows:

“For a group  $G$ , the quotient group  $G/H$  is the set of cosets  $Hx$  ( $x \in G$ ) of the normal subgroup  $H$  of  $G$ , with multiplication given by  $Hx_1Hx_2 = Hx_1x_2$  ( $x_1, x_2 \in G$ ).”

According to Nicholson (1993, p. 69), this definition is called the "standard" definition because:

“First, this definition makes use only of the elements of the group  $G$  itself, with these elements combined in a particular way. We do not have to use any concepts “outside” the group. Second, the definition does not depend on representing  $G$  in any one way: it is “abstract” and can be applied to any group.”

## 3. Epistemological analysis of the history of the concept of quotient group

Although the group concept was once considered the basis for group research, it was a concept that group theorists had not known before, even though it was present in their works. The way in which the group concept was discovered and developed contributed by most to the 7 mathematicians: Évariste Galois, Enrico Betti, Camille Jordan, Richard Dedekind, Walther von Dyck, Ferdinand Georg Frobenius and Otto Ludwig Hölder.

We will analyze the formation and development of the quotient group concept until the “standard definition” appears and is used today.

### 3.1. The birth of the concept of quotient group

We can not point out by whom or at which time the discovery and development of the concept of quotient group were made. Like most mathematical ideas, the development of this concept, from its primitive form to its popularity in the mathematical community, occurred for a long time and was contributed by many mathematicians.

“The opinion of modern commentators on this subject is quite multifaceted: Bourbaki argues that “Jordan introduced the concept”, while Van Der Waerden said, “The modern understanding of the quotient group is attributed to Hölder and that Jordan just defined it implicitly”. Wussing notes that “the idea of a quotient group is derived from the abstract group concept”, and he later notes that “Hölder put an abstract definition of a group at the beginning of his 1889 paper”, which shows that Wussing considered Hölder to be the first to introduce the concept of quotient group.” (Nicholson, 1993, p. 84)

The implicit presence of the concept of quotient group was discovered in the Galois theory of algebraic equation in 1832 by a notable idea (similar to the idea of the concept of quotient group) – “the way in which the group of such an auxiliary equation arises from the group of the original equation”. Galois, however, had no idea about the quotient group.

From 1852 to 1855, Betti was seeking to explain “the way in which the group of an auxiliary equation is related to the group of the original equation” by Galois from the viewpoint of substitution group theory. Betti tried to explain the fact that a normal subgroup of a group creates another group so-called a quotient group, and he sought to do so in substitution group theory. Despite substantial achievement, Betti seemed to only understand some of the ideas behind the concept of quotient group.

In the years 1870 - 1873, the works of Jordan identified the multiplication in the quotient group structure. The abstraction and symbols in the representative system of the notion of “moldulo” appeared in the Jordan’s proof. However, Jordan’s definition of a quotient group was only considered on the groups of substitutions or transformations (finite group), so it did not satisfy the two criteria of the “standard definition” of the concept of quotient group.

The concept of quotient group continued to be found in Dedekind’s work. The concept of the “Dedekind Cut” was introduced by Dedekind himself in 1858 and in the years 1855–1858. He explored the theory of the concept of quotient group: He expressed  $M$  in terms of  $N$  and its “cosets” and stated that a “composition” of cosets could be defined and could form a group. Although Dedekind did not name the quotient group, the idea of “cosets” and “composition” influenced the development and formation of the “standard definition” of the concept of quotient group.

In 1882, Dyck’s paper, “Gruppentheoretische Studien”, began with the construction of any  $G$  group with the elements  $A_1, A_2, \dots, A_m$ . Dyck was concerned with exploring the way in which any group  $\bar{G}$  generated by elements  $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_m$ . He proved that the relation between  $G$  and  $\bar{G}$  can be established by homomorphism  $A_i \rightarrow \bar{A}_i$  ( $\bar{G}$  is a homomorphic image of  $G$ ). In his justification, he provided two cases for consideration: The first case is when the groups are isomorphic; the second occurs when any element of  $\bar{G}$  corresponds to infinitely many elements of  $G$ . In the second case, he finds the elements of  $G$  corresponding to the unit of  $\bar{G}$  and shows that they form a normal subgroup  $H$  of  $G$ .

Although Dyck does not provide any explanation for the quotient group, the  $\bar{G}$  group generated by any  $G$  group partially satisfies the two criteria of the “standard definition”.

In the 1880s, Frobenius introduced the idea of equivalence between the elements of the group. His joint paper with Stickelberger on abelian groups (1879) contained an early formulation of this idea. In this paper, the fundamental theorem of abelian groups was proved by using equivalence classes with respect to subgroups of a given abelian group, and Frobenius mentioned only of a set of “elements” and a well-defined binary operation on the set, and gave other definite conditions for an abelian group. Apparently Frobenius thought of the equivalence class as one element, which was an important step towards perfecting the quotient group.

The final stage in the development of the concept of quotient group was associated with Hölder's paper in the journal “Mathematische Annalen” entitled “Reduction of an arbitrary algebraic equation to a chain of equations” (1889). Hölder (1889, p. 30) wrote:

“This theory of the composition factors must however be furthered, in the sense that the factors are to be understood as groups.

It will be shown in the next paragraph that through the relationship of a group to a normal subgroup contained in it, a new group of (usually) other operations is always defined. This latter group is completely determined from the abstract standpoint, which disregards the nature of the operations...”

Hölder then defines “quotient group” of a group by a normal subgroup. He showed how the elements of a group  $G$  could be divided amongst the cosets of any subgroup  $H$  and proved that if  $H$  is normal, the multiplication of any two elements from two cosets will always give an element in one and the same coset. He continued (1889, p. 31):

“In this way one obtains new operations, which likewise form a group. It is this well-defined group that is to be brought into consideration. One could call it the quotient of the groups  $G$  and  $H$  and it will from now on be denoted by  $G/H$ .”

Thus in this paper, the concept of quotient group has been systematically and explicitly defined. Hölder introduced the terms “Quotient”, “Factorgruppe” and the notation  $G/H$ : the terms have remained but the notation has been combined with Jordan's to produce our modern  $G/H$ . So, according to our viewpoint, 1890 can be seen as a milestone marking the birth of the concept of quotient group.

### **3.2. The conceptions of the formation of the concept of quotient group**

#### **3.2.1. Galois's arithmetic conception**

The primitive idea of the concept of quotient groups was found in the Galois theory of algebraic equations. Galois made it explicit for the first time the concepts of group, the normality of a subgroup and he discussed how a given equation can be solved by investigating the structure of its associated group. He used “le groupe” to refer to a set of

arrangements of the roots of the equation rather than a set of permutations of these arrangements. He, however, understood that it is the permutations which have the “group structure”. Thus he wrote:

“... when the group of an equation admits a proper decomposition, in such a way that it divides into  $M$  groups of  $N$  permutations, one will be able to solve the given equation by means of two equations: one of them will have a group of  $M$  permutations, the other a group of  $N$  permutations.” (Nicholson, 1993, p.70)

In modern terms, this remark states that if the group  $G$  of an equation has a normal subgroup  $H$ , the equation can be solved by means of two equations whose groups we know as  $G/H$  and  $H$ . Since Galois had no concept of quotient group, the group that now is called  $G/H$  for was to him just “the group associated with its auxiliary equation.”

### 3.2.2. Betti's arithmetic conception

In 1849, Betti was aware of the advances in substitution group theory as set out in Serret's textbook. Thus, in 1852, in his first commentary on Galois published papers, Betti was seeking to explain Galois's work from the viewpoint of substitution group theory. He focused on the way in which the group of an auxiliary equation is related to the group of the original equation. The first half of his paper aimed at the treatment of this question from a purely group-theoretic viewpoint. That is, Betti was searching for a way to explain the fact that a normal subgroup of a group gave rise to another group which we now understand as a quotient group and he was seeking to do this within substitution group theory.

### 3.2.3. Jordan's arithmetic conception

The idea of “modulo” calculation on a normal subgroup is indeed the idea that gave birth to the Jordan's concept of quotient group. In his approach, there is a remarkable similarity to Gauss's work on arithmetic congruences in 1801. Jordan employed the symbol “ $\equiv$ ” that Gauss introduced to denote congruence.

In 1873, when Jordan extended some results of Mathieu on the limit of transitivity of groups by way of Sylow's Theorems. He used the idea of congruence of group elements to produce a quotient group structure.

### 3.2.4. Dedekind's abstract conception

Dedekind appears to have understood the role of equivalence at a much earlier period, in particular in his work during the years 1855–1858. Dedekind explored the concept of homomorphism in a section entitled “Äquivalenz von Gruppen”. He formed a homomorphic image  $M_I$  of a group  $M$  by letting each element  $\theta$  of  $M$  “correspond” to an element  $\theta_I$  of  $M_I$ , with certain conditions which we now recognize as the conditions for homomorphism. He proved that  $M_I$  is a group and that those elements of  $M$  which “correspond” to the identity in  $M_I$  form a subgroup  $N$  of  $M$ . He went on to discover the concept of quotient group:

He expressed  $M$  in terms of  $N$  and its cosets and stated that a “composition” of cosets can be defined and that in this way the cosets (he referred to them simply as “Komplexe”) form a group. There is a correspondence between the cosets and the elements of  $M_I$  such that to each coset corresponds one element of  $M_I$ , and to each element of  $M_I$  corresponds one coset. We would now say that the group  $M_I$  and the group of cosets are isomorphic. Dedekind gave no name either to the concept of homomorphism or to that of quotient group.

### 3.2.5. Dyck's abstract conception

In 1882, Dyck's paper, “Gruppentheoretische Studien” began with the abstraction of any  $G$  group with the elements  $A_1, A_2, \dots, A_m$ . In Section 4, Dyck was interested in exploring the way in which any group  $\bar{G}$  generated by elements  $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_m$ , is defined as “some predetermined process”, which is related to the original  $G$  group.

Dyck's notation here already suggests a hidden assumption that  $\bar{G}$  is a homomorphic image of  $G$  under the homomorphism taking  $A_i \rightarrow \bar{A}_i$ . Dyck showed that there are two cases to consider: The first case is when the groups are isomorphic; the second occurs when any element of  $\bar{G}$  corresponds to infinitely many elements of  $G$ . In the second case, he finds the elements of  $G$  corresponding to the unit of  $\bar{G}$  and show that they form a normal subgroup  $H$  of  $G$ . He investigated the structure of this normal subgroup and then stated that the relationship between  $G$  and  $\bar{G}$  can be set out:

“The relationship of isomorphism between the groups  $G$  and  $\bar{G}$  splits the group  $G$  into two factors: into the group of substitutions which are different when written in terms of the substitutions  $\bar{A}_i$ , that is, the group  $\bar{G}$  itself, and into the group  $H$  of those substitutions which, when written in terms of the substitutions of  $\bar{G}$ , are equivalent to the identity. The latter group  $H$  is then contained as a normal subgroup in  $G$  and comes from it “by adjunction of  $G$ .” (Nicholson, 1993, p. 77)

The phrase “durch Adjunction von  $\bar{G}$ ” seems to have been borrowed from Galois theory, since one reduces the Galois group of an equation to a normal subgroup by adjoining elements to the field, and these elements are the roots of an equation whose Galois group is  $\bar{G}$ . So the group  $G$  can be thought of as having two factors: its normal subgroup  $H$  and the homomorphic image  $\bar{G}$  which now, of course, we also know as the quotient group  $G/H$ .

### 3.2.6. Frobenius's abstract conception

In the late 1870s and the 1880s, Frobenius was also led to consider the idea of equivalence of group elements. His joint paper with Stickelberger on abelian groups (1879) contains an early formulation of this idea. In this paper the fundamental theorem of abelian groups was proved by using equivalence classes with respect to subgroups of a given abelian group (and since it is abelian all subgroups are normal). This viewpoint was borrowed from an paper of Kronecker's of 1870.

Frobenius later developed a new proof of Sylow's Theorems, which was published in 1887. In this short paper, Frobenius followed Jordan's approach to the concept of quotient group but in the setting of abstract group theory. Frobenius proved that if the order of a group is divisible by  $p^v$  where  $p$  is prime, then the group has a subgroup of order  $p^v$ .

In another paper dealing with congruence of group elements which was published soon after that on Sylow's theorem, Frobenius used the ideas of equivalence relations and equivalence classes to define double cosets, again citing Kronecker and Jordan for the concept of equivalence of group elements.

He investigated the properties which such cosets possess, including the fact that the number of equivalence classes does not change when a common normal subgroup  $N$  is factored out. His use of the concept of quotient group here follows Jordan's definition and notation of 1873, except the statement that each set of elements congruent mod  $N$  is to be considered as one element. Then these "complexen Elemente" form a group, the quotient group, by the normal subgroup  $N$ .

The fact that Frobenius was able to think of each congruence class as one element was an important breakthrough because his understanding of the abstract approach allowed for it. Later, in a paper on finite groups in 1895, Frobenius cited both Jordan and Hölder when making use of quotient groups and attributed the definition to them equally.

### 3.2.7. Hölder's abstract conception

This was the final stage in the development of the concept of quotient group with Hölder's work in the journal "Mathematische Annalen", entitled "Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen" (1889).

The questions which Hölder wished to answer here are those prompted by taking a fresh look at Galois theory in the light of abstract group theory. Which groups correspond to the "auxiliary equations"? To what extent are these groups defined? How many are there? The natural way to answer these questions is to employ the concept of quotient group.

In the introduction Hölder discussed the simple groups arising from a composition series, which he named "Factorgruppen" and noted that this concept of "Factorgruppe" is "a group-theoretic idea that has until now not been adequately appreciated". He stated that he would set out on only the most elementary group-theoretic ideas in the discussion that followed. It seems that Hölder did not consider the concept of quotient group either a new or a difficult one.

The first part of the paper is a group-theoretic section. Hölder gave axioms for a finite group and mentioned normal subgroups and composition series. He then talked about "Factoren der Composition": a "Factor der Composition" is the index of a group in a composition series in the preceding group of the series, as defined by Jordan. In modern terms these are the orders of the composition factors.

Hölder came to define the “Quotient” of a group by a normal subgroup. He showed how the elements of a group  $G$  can be divided amongst the cosets of any subgroup  $H$  and proved that if  $H$  is normal, the multiplication of any two elements from two cosets will always give an element in one and the same coset. In other words, in this case we can define multiplication of cosets and it is well-defined.

The next paragraphs show that this concept can be expressed in terms of equivalence of group elements. We call two elements of  $G$  equivalent if they can be transformed into one another by multiplication with an element of the normal subgroup  $H$ . Then the equivalence classes will form a group.

Thus in this paper the concept of quotient group was systematically and explicitly defined and its previously implicit appearance in Galois theory was recognized. Hölder introduced the terms “Quotient” and “Factorgruppe” and the notation  $G/H$ : the terms have remained but the notation has been combined with Jordan's to produce our modern  $G/H$ .

Here, the term “Factorgruppe” is used by Holder to refer to simple quotient groups deriving from a composition series because he believes the quotient groups can analyse the original groups, so the quotient groups are considered as “prime factors” in arithmetic.

**3.3. Epistemological characteristics of the concept of quotient groups**

After analysing of the historical process to the formation of the concept of quotient group based on references: Bourbaki (1969), Wussing (1984), Nicholson (1993), Burton (2011), we find the epistemologicals characteristics of the concept of quotient groups following (Table 1).

**Table 1.** The epistemologicals characteristics of the concept of quotient expressed in the work of seven mathematicians

Mathematicians	Galois	Betti	Jordan	Dede-kind	Dyck	Frobe-nius	Hölder	
Mechanism of the concept	Outil	Object	Object	Object	Object	Object	Object	
Expression Mode of concept	ProM	ProM	ParaM	ParaM	ParaM	ParaM	Math	
Characteristics	Intrinsic	✓	✓	✓	✓	✓	✓	
	Specific / Abstract	⊕	⊕	⊕	◇	◇	◇	
	Multi – access	a	b	c	d	b, e	f	g
	Global	✓	✓	✓	✓	✓	✓	✓
	Structural	×	✓	✓	✓	✓	✓	✓
	Similar	×	×	✓	×	×	×	✓
	Axiomatical	×	×	×	×	×	×	✓

Glossary of the expression modes of concept:

According to Chevallard (1991), a mathematical concept can be expressed in three forms:

- *Protomathematic* (ProM): no name, no definition, working as an implicit tool.
- *Paramathematic* (ParaM): named, no definition, being a tool of mathematical activity.
- *Mathematic* (Math): being both a research object and a tool used to solve problems.

Glossary of the characteristics of the concept of quotient group:

- *Intrinsic* characteristic: The quotient group is derived from the elements of the original group.
- *Specific/abstract* characteristics: associated with the investigation of solutions of algebraic equations (specific -  $\oplus$ ); associated with the theory of groups of substitutions or transformations (specific -  $\oplus$ ); construction of general quotient group (abstract -  $\diamond$ ).
- *Multi-access* characteristic: by the set of arrangements of the roots of the algebraic equations (a); by cosets (b); by congruence (c); by equivalence relation and homomorphism (d); by group isomorphism (e); by cyclic subgroups (f), by normal subgroups (g)
- *Global* characteristic: construction of the quotient group from a subset of the elements of the original group.
- *Structural* characteristic: the quotient groups include equivalence relation, quotient set, cosets, equivalence classes, normal subgroup, homomorphism, isomorphism.
- *Similar* characteristic: using the concept of congruence in arithmetic to construct the quotient group structure.
- *Axiomatical* characteristic: defining the concept of quotient group by axiomatic system.

### 3.4. Identified epistemological obstacles

Based on the results of the analysis of the history of quotient group formation, we identify three epistemological obstacles of the quotient group:

- *Abstract obstacle* from the abstraction of the concept of quotient group by symbolic representative system. This obstacle generates the difficulties that students face when they transfer from research on sets of specific numbers (represented objects) to research on symbol systems (representative objects).
- *Structural obstacle* in the structuralization of the quotient group into classes by equivalence relation.

- *Intrinsic obstacle* in the construction of the quotient group from elements of the original group.

These three obstacles are associated with the history of formation of the quotient group and can become obstacles to students when approaching the quotient group.

### 3.5. Hypothesis

With three main difficulties identified during the student interviews in section 1.2:

- the distinction between the elements of the quotient group and those of the original group;
- the comprehension of nature of the elements and that of the operations of quotient group;
- the realization of fundamental factors in the building of a quotient group, and from the results of the epistemological analysis in sections 3.2 and 3.3, we construct the hypothesis **H** of students' difficulties when first approaching the concept of quotient groups as follows:

*The above three difficulties can be identified in most students when approaching the concept of quotient groups for the first time and these difficulties derive from three epistemological obstacles: intrinsic obstacle, abstract obstacles and structural obstacle of the quotient group.*

## 4. Conclusion

Historical epistemological analysis has demonstrated that concept of quotient group is influenced by many other mathematical concepts such as cosets, equivalence classes, equivalence set, normal groups, homomorphisms and isomorphisms. In particular, the evolution of the concept of quotient group is closely associated with the thriving of equivalence theory and the process of abstraction of mathematics.

To validate the hypotheses mentioned in Section 3.5, in the next study we will conduct an experiment to elaborate on students' three difficulties in approach of the concept of quotient group and analyze the causes underlying these difficulties. The results of the study will be covered in detail in another article.

❖ **Conflict of Interest:** Author have no conflict of interest to declare.

## REFERENCES

- Artigue, M. (1991). Epistémologie et Didactique. *Recherche en Didactique des Mathématiques*, Vol.10, No.3, 241-286.
- Bessot, A., Comiti, C., Le, T. H. C. & Le, V. T. (2009). *Fundamental elements of Didactic of Mathematics*. Ho Chi Minh City: National University.
- Bourbaki, N. (1969). *Éléments d'Histoire des Mathématiques*. Hermann.
- Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en didactique des mathématiques*, 4(2), 165-198.
- Burton, D. M. (2011). *The History of Mathematics: An Introduction*. Seventh Edition, McGraw-Hill, Inc.
- Cayley, A. (1889). *The Collected Mathematical Papers*, Vol. 2, Cambridge.
- Chevallard, Y. (1991). *La transposition didactique du savoir savant au savoir enseigné*. Grenoble: La pensée Sauvage.
- De Serres, M. & Groleau, J. D. (1997). *Mathématiques et langages*. Montréal: Collège Jean-de-Brébeuf.
- Dong Nai University. (2013). *Tertiary Education Curriculum and Training Program: Mathematics Pedagogy*.
- Duval, R. (1993). Registre de représentation sémiotiques et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, Vol.5, 37-65, IREM de Strasbourg.
- Duval, R. (1995). *Semiosis et pensée humaine: Registres sémiotique et apprentissages intellectuels*. Berne: Peter Lang.
- Frobenius, G. & Stickelberger, L. (1879). Über Gruppen von vertauschbaren Elementen. *Journal für die reine und angewandte Mathematik* 86, 217-262; Frobenius, *Gesammelte Abhandlungen*, J.-P. Serre, Ed., Vol. I, pp. 545-590. Berlin: Springer-Verlag, 1968. Page references are to the *Journal* and *Abhandlungen*, respectively.
- Herstein, I. N. (1975). *Topics in algebra*, 2nd ed., New York: Wiley.
- Ho Chi Minh City Pedagogical University. (2016). *Tertiary Education Curriculum, Faculty: Mathematics Pedagogy*.
- Hoang, X. S. & Tran, P. D. (2003). *General Algebra*. Ha Noi: Pedagogical University.
- Hölder, O. (1889). Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen. *Mathematische Annalen*, No. 34, 26-56.
- Jordan, C. (1961). *Oeuvres de Camille Jordan*, J. Dieudonné, Ed., Vol. I. Paris: Gauthier-Villars.
- Katz, V. J. (2009). *A History of Mathematics – An Introduction*. 3rd Edition, Pearson Education, Inc.
- Kleiner, I. (1986). *The Evolution of Group Theory: A Brief Survey*. *Mathematics Magazine*, Vol.59, No.4, pp.195-215.

- Kleiner, I. (2007). *A History of Abstract Algebra*. Secaucus, United States: Birkhauser Boston Inc.
- Lajoie, C. & Mura R. (2004). Difficultés liées à l'apprentissage des concepts de sous-groupe normal et de groupe quotient. *Recherches en didactique des mathématiques*, Vol.24, N<sup>o</sup>1, 45-80.
- Lajoie, C. (2009). *Un groupe en quête de théorie: un problème pour les étudiants en maths!*. Montréal: Éditions Bande didactique.
- Le, T. H. C. (2017). The necessity of epistemological analysis for research on teaching activities and teacher training. *6th International Seminar on Didactic Mathematics*, Ho Chi Minh City Pedagogical University.
- Le, V. T. (2003). A new perspective on the process of teaching the concept of mathematics. *Journal of Education*, No. 64, Ha noi.
- Lewis, J. (2013). *An Analyse of Students' Difficulties in Learning Group Theory*. Thesis in The Department of Mathematics and Statistics, Concordia University, Montreal, Quebec, Canada.
- Macdonald, I. D. (1968). *The theory of groups*. Oxford: Oxford Univ. Press.
- Mili, R. I. (2016). *Identification d'obstacles et de difficultés inhérents à l'apprentissage de l'algèbre abstraite*. Mémoire présenté à la Faculté des études supérieures en vue de l'obtention du grade de maîtrise ès arts en sciences de l'éducation, option didactique, Université de Montréal.
- Nguyen, A. Q. & Nguyen, T. V. K. (2017). An epistemological analysis in teaching advanced algebra: the case of group. *6th International Seminar on Didactic Mathematics*, Ho Chi Minh City Pedagogical University.
- Nguyen, A. Q., Nguyen, T. T. T. (2017). The contributions of epistemological analysis for teaching linear algebra: the case of vector space. *6th International Seminar on Didactic Mathematics*, Ho Chi Minh City Pedagogical University.
- Nicholson, J. (1993). The development and understanding of the concept of quotient group, *Historia Mathematica*, 20, 68 - 88.
- Rose, J. S. (1978). *A course on group theory*. Cambridge: Cambridge Univ. Press.
- SaiGon University. (2016). *Tertiary Education Curriculum, Faculty: Mathematics Pedagogy*.
- Wussing, H. (1984). *The Genesis of the Abstract Group Concept*. Cambridge, Massachusetts: M.I.T. Press. (Translation by Abe Shenitzer of the 1969 German Edition).